

Kindergarten Quantum Mechanics

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(or [Google](#) ‘‘Bob Coecke’’)

THE CHALLENGE

Why did discovering quantum teleportation take 60 years?

Claim: **bad formalism** since 'too low level' cf.

$$\frac{\text{“GOOD QM”}}{\text{von Neumann QM}} \approx \frac{\text{HIGH-LEVEL language}}{\text{low-level language}}$$

Wouldn't it be nice to have a such a 'good' formalism, in which discovering teleportation would be trivial?

Claim: **it exists!** And I'll present it to you.

Isn't it absurdly abstract coming from you guys?

Claim: It could be taught in **kindergarten!**

THE APPROACH

1. **Analyse** quantum compoundness.

⇒ A notion of **quantum information-flow** emerges.

- **Physical Traces**. Abramsky & Coecke (2003) CTCS'02; cs/0207057
- **The Logic of Entanglement**. Coecke (2003) PRG-RR; quant-ph/0402014
- **Quantum Information-flow, Concretely, and Axiomatically**. quant-ph/0506132

2. **Axiomatize** quantum compoundness.

⇒ ... full **quantum mechanics** emerges!

- **A Categorical Semantics of Quantum Protocols**. Abramsky & Coecke (2004) IEEE-LICS'04; quant-ph/0402130
- **Abstract Physical Traces**. Abramsky & Coecke (2005) TAC'05.

⇒ ... & **quantum logic** ... & **open systems/CPM's!**

- **De-linearizing Linearity I: Projective Quantum Axiomatics from SCC**. Coecke (2005) QPL'05; quant-ph/0506134.
- **†-CCC's and Completely Positive Maps**. Selinger (2005) QPL'05.

EXPLICIT OPERATIONALISM

Primitive data are **processes/operations** f, g, h, \dots which are **typed** as $A \rightarrow B, B \rightarrow C, A \rightarrow A, \dots$ where A, B, C, \dots are **kinds/names** of **systems**.

Sequential composition is a primitive connective on processes/operations cf.

$$f \circ g : A \rightarrow C \quad \text{for} \quad f : A \rightarrow \underline{B} \ \& \ g : \underline{B} \rightarrow C$$

Parallel composition is a primitive connective both on systems and processes/operations cf.

$$f \otimes g : A \otimes C \rightarrow B \otimes D \quad \text{for} \quad f : A \rightarrow B \ \& \ g : C \rightarrow D$$

NO DOGMAS nor TABOOS!

Do you want ...

- states to be **ontological** or **empirical**?
- **vectorial**, **projective**, **POVM-/CPM-/open system-style**?
- **hidden variables**, **quantum potential**, **contextuality**, **(non-)locality**, **Bayesianism**, ... ?

The bulk of the developments ignores these choices, but, they can be implemented formally since we both have

- **great axiomatic freedom**
- **great expressiveness**

CATEGORY THEORY!

Audience: “Seriously, you don’t expect us to learn that?”

Bob: “**No! Of course not!**”

“We are gonna go far back in time, ...
to the time you were all still at kindergarten, ...”

“**We’re gonna draw pictures!**”

The sheer magic of the kind of category theory we need here is that **it formally justifies its own formal absence.**

A NEW FORMALISM

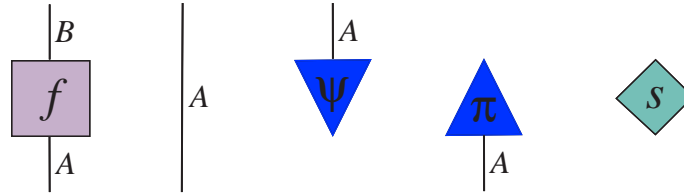
Language and calculus: **purely graphical**

Behind the scene: **categorical algebra**

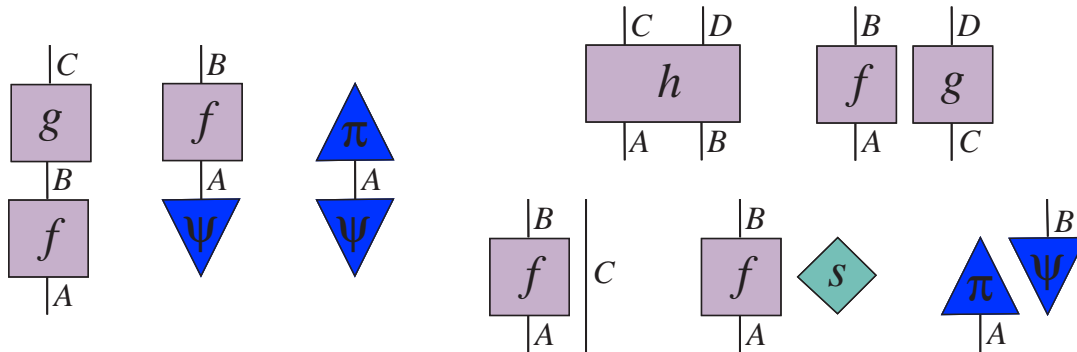
Concrete model: **Hilbert space QM, ...
and also many others, ...**

Not assumed: some **number field**, any kind of **matrix calculus**, **vectors** and **sums** thereof, **elements** of objects/types (cf. state space) and corresponding **mappings**, ...

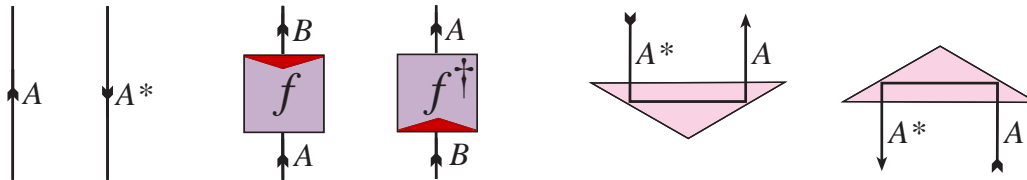
Primitive data:



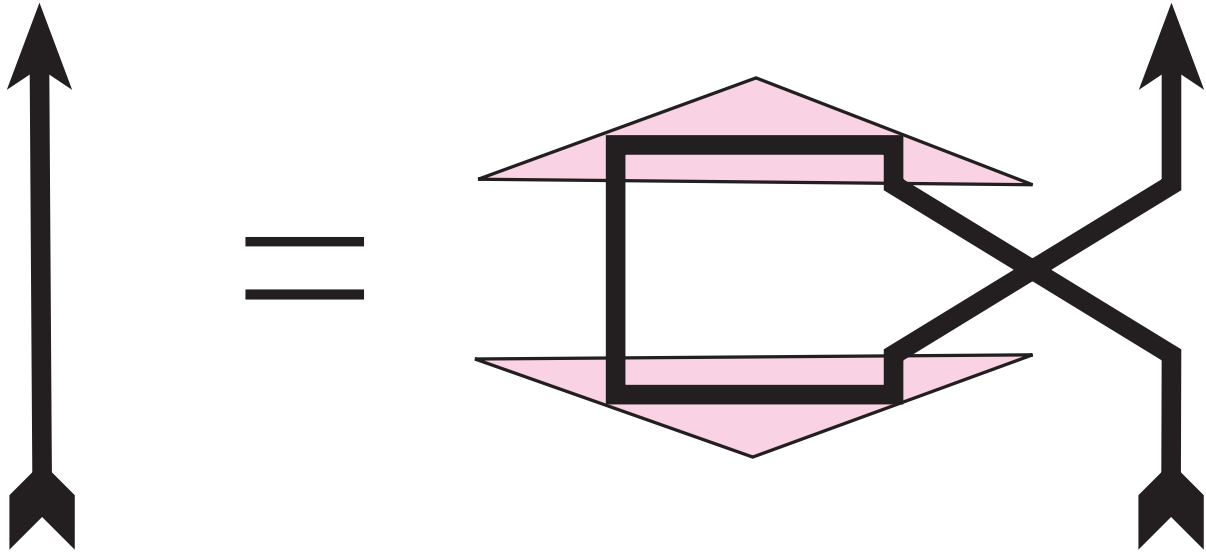
Sequential and parallel composition:



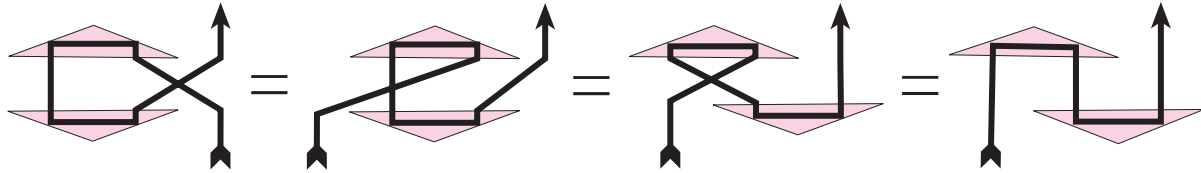
Duals, adjoints and EPR-states:



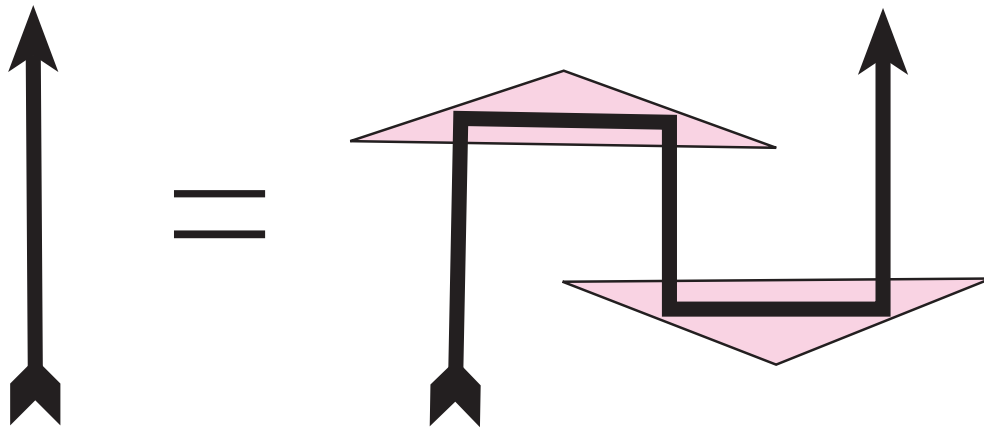
THE SOLE AXIOM



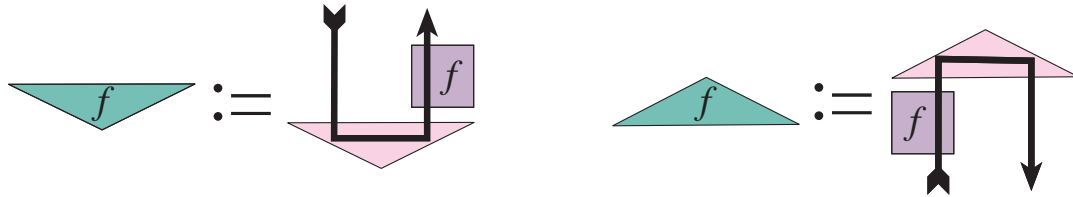
Since



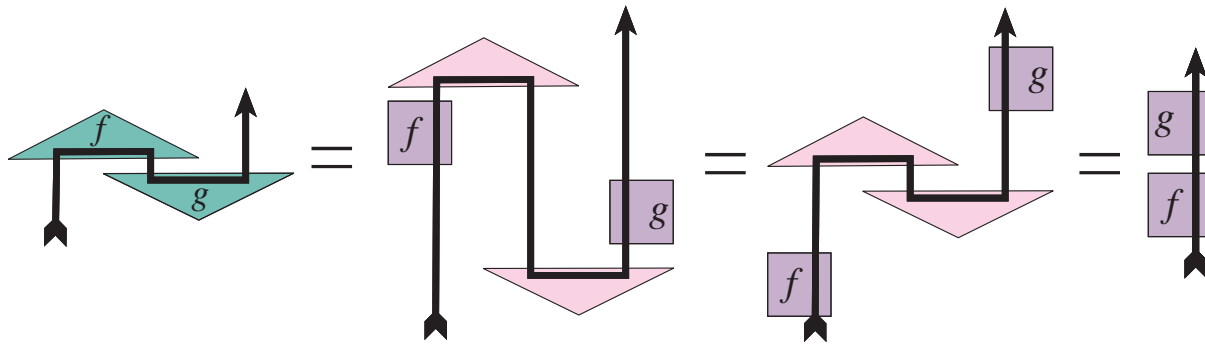
the axiom is equivalent to



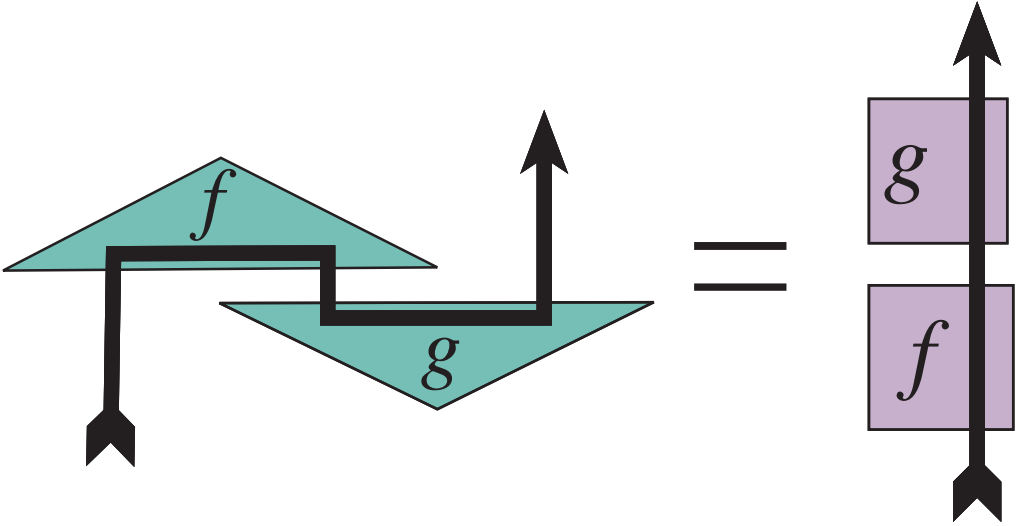
When setting



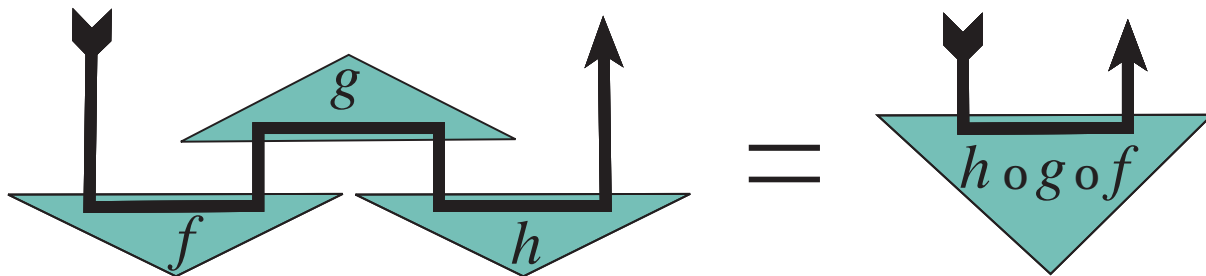
we obtain



COMPOSITIONALITY



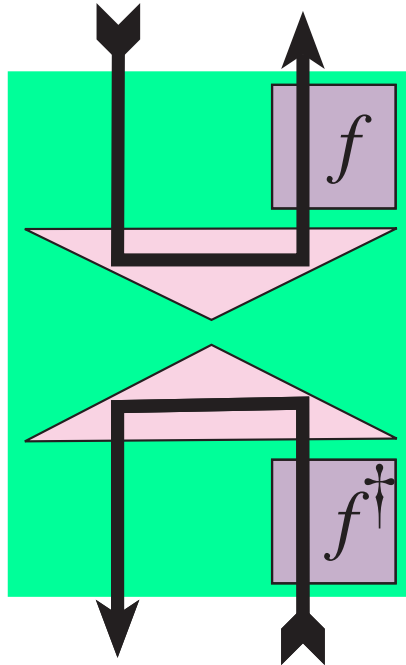
COMPOSITIONALITY bis



We define bipartite projectors as

$$P_f : A^* \otimes B \rightarrow A^* \otimes B$$

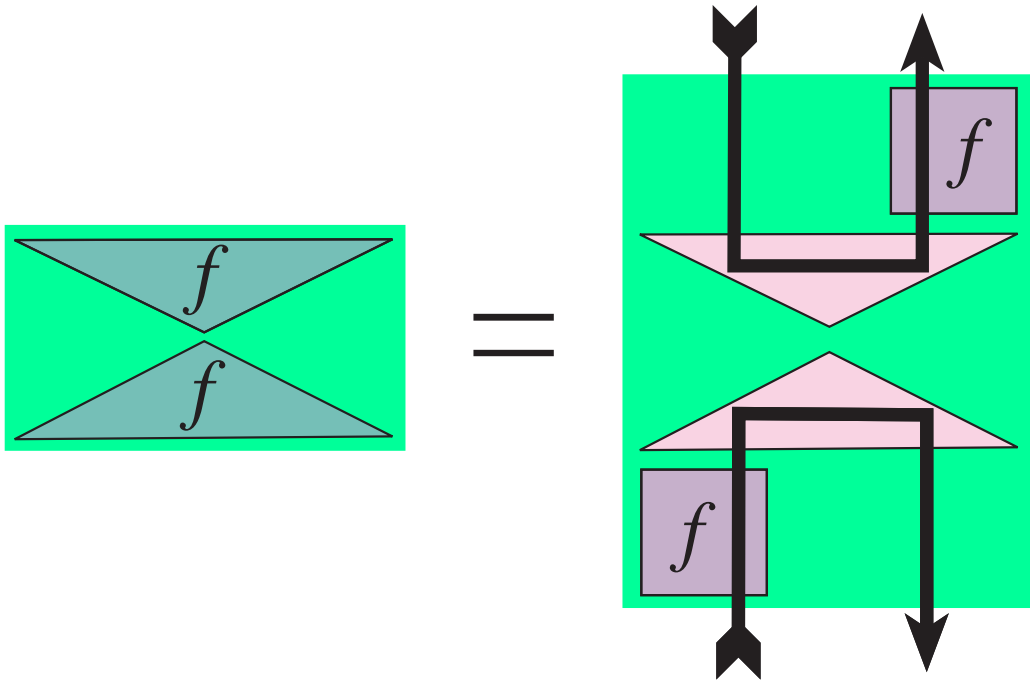
as



that is, approximately, as

$$P_f : A \otimes B^* \rightarrow A^* \otimes B$$

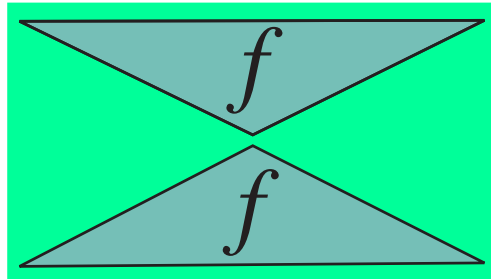
as



The concepts of bipartite state

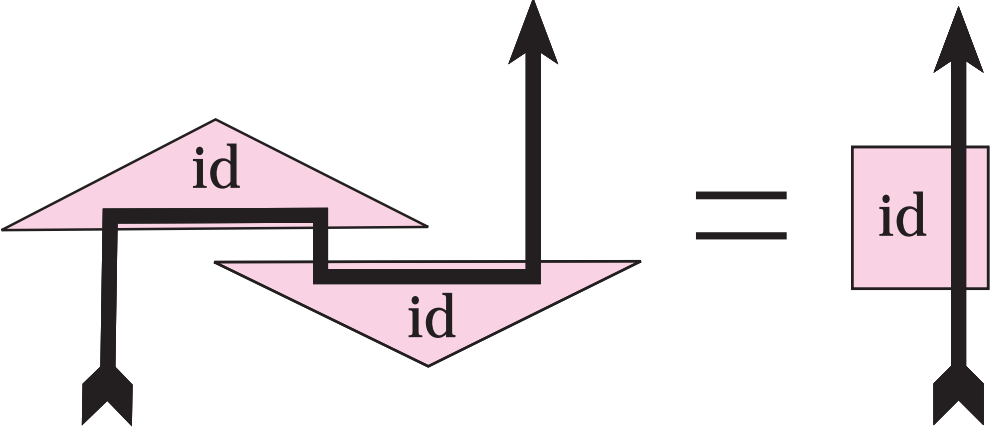


and of bipartite projector



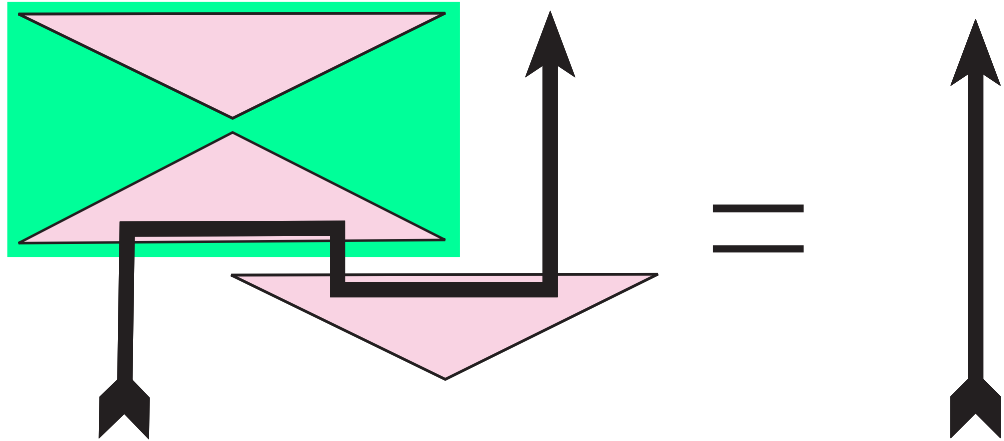
yield the following corrolaries ...

$\frac{1}{4}$ th-TELEPORTATION



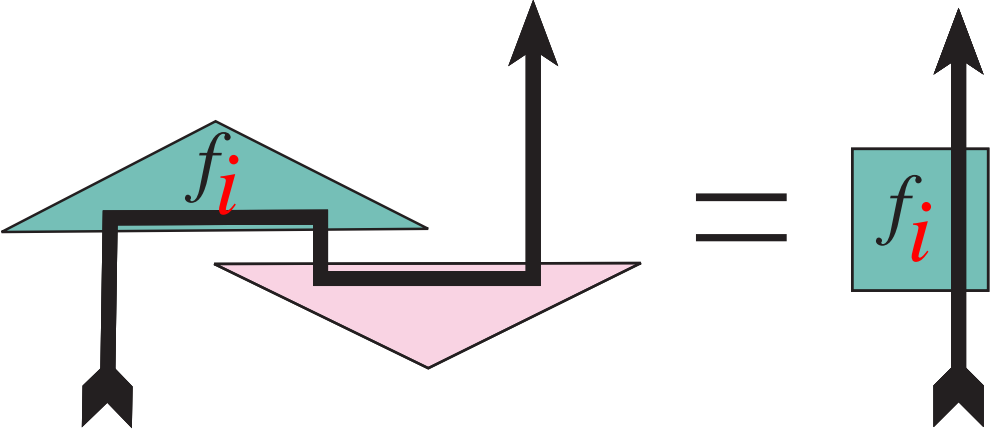
since $\text{id} \circ \text{id} = \text{id}$

$\frac{1}{4}$ th-TELEPORTATION



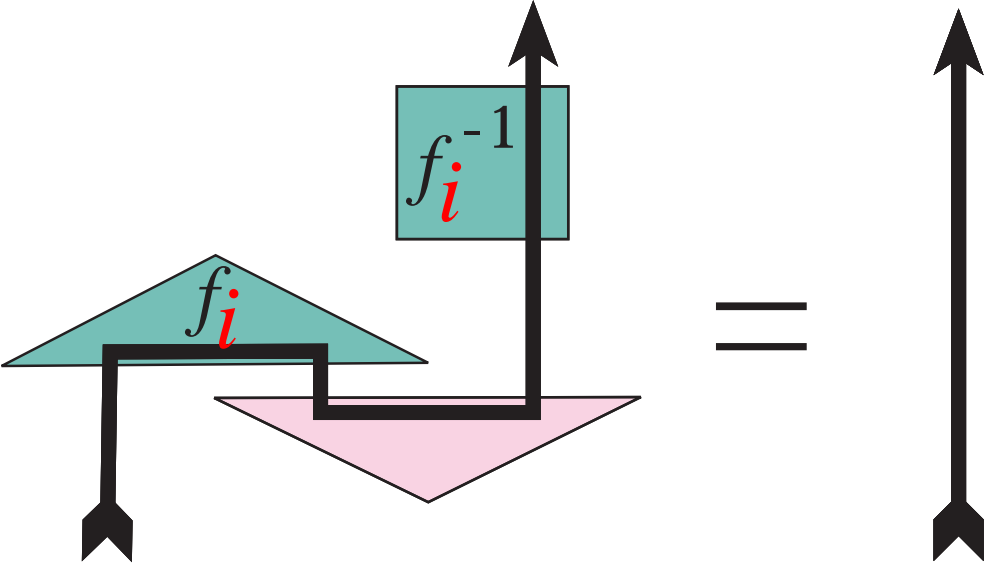
since $\text{id} \circ \text{id} = \text{id}$

FULL TELEPORTATION



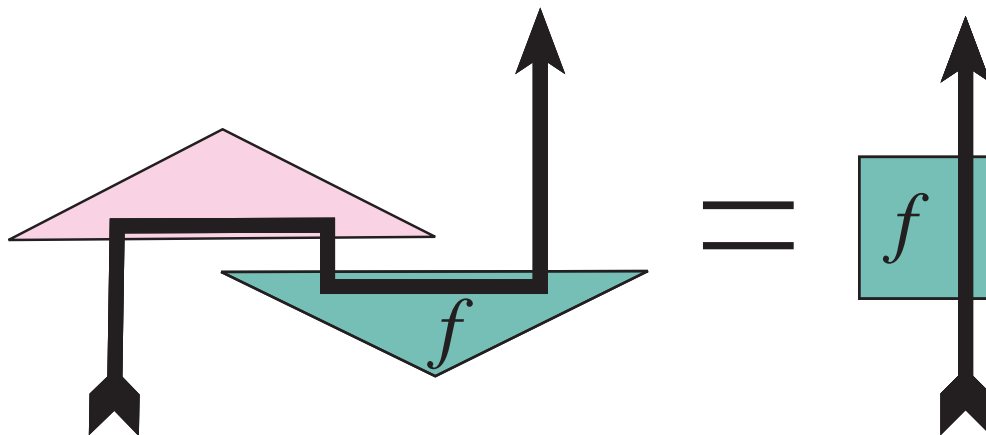
for $1 \leq i \leq 4$

FULL TELEPORTATION



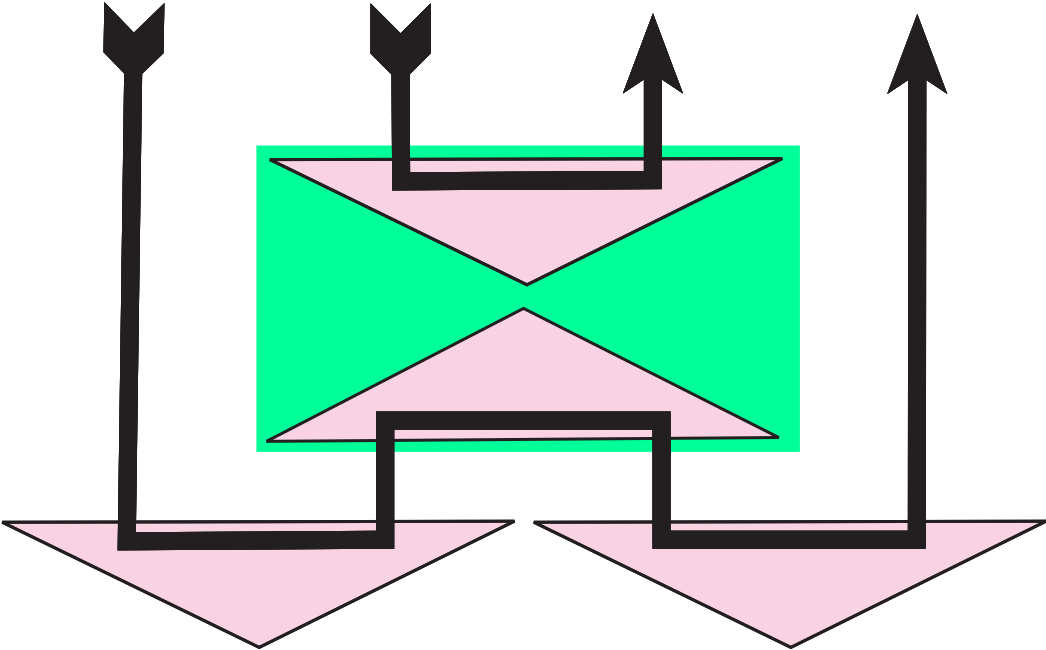
for $1 \leq i \leq 4$

LOGIC GATE TELEPORTATION



since $f \circ \text{id} = f$

ENTANGLEMENT SWAPPING



HILBERT SPACE QM

f : $\mathcal{H}_1 \rightarrow \mathcal{H}_2$ is a linear map

Ψ : $\mathbb{C} \rightarrow \mathcal{H}$ cf. $\psi(1) \in \mathcal{H}$

s : $\mathbb{C} \rightarrow \mathbb{C}$ cf. $s(1) \in \mathbb{C}$

\mathcal{H}^* := conjugate Hilbert space of \mathcal{H}

f^\dagger := linear adjoint of f

$$\begin{array}{c} \Psi \\ \downarrow \end{array} = |\psi\rangle \quad \begin{array}{c} \uparrow \pi \\ \uparrow \end{array} = \langle \phi | \quad \text{for } \pi := \phi^\dagger \quad \begin{array}{c} \uparrow \pi \\ \downarrow \Psi \end{array} = \langle \phi | \psi \rangle$$

HILBERT SPACE QM

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EPR-states and their adjoints:

$$\begin{array}{c} \uparrow \\ \text{---} \\ \downarrow \end{array} : \mathbb{C} \rightarrow \mathcal{H}^* \otimes \mathcal{H} :: 1 \mapsto \left| \sum_i e_i \otimes e_i \right\rangle$$

$$\begin{array}{c} \text{---} \\ \uparrow \\ \downarrow \end{array} : \mathcal{H}^* \otimes \mathcal{H} \rightarrow \mathbb{C} :: \Phi \mapsto \left\langle \sum_i e_i \otimes e_i \mid \Phi \right\rangle$$

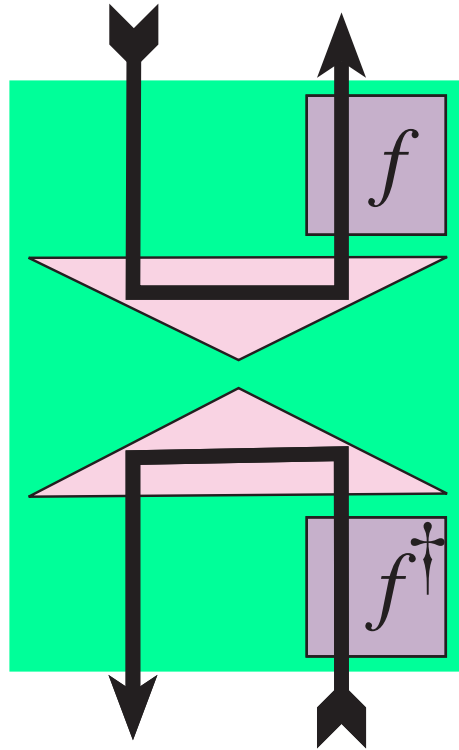
$$:: \phi_1 \otimes \phi_2 \mapsto \langle \phi_1 \mid \phi_2 \rangle$$

We verify the axiom:

$$\begin{array}{c} \uparrow \\ \text{---} \\ \downarrow \end{array} = (-) \otimes \left(\sum_i e_i \otimes e_i \right) = \sum_i (- \otimes e_i) \otimes e_i$$

$$\begin{array}{c} \text{---} \\ \uparrow \\ \downarrow \end{array} = \sum_i \langle - \mid e_i \rangle \cdot e_i = \text{id}$$

Exercise. Verify that in Hilbert space bipartite projectors on one-dimensional subspaces indeed factor as



A key role is played by

$$\mathcal{H}_1^* \otimes \mathcal{H}_2 \simeq \mathcal{H}_1 \rightarrow \mathcal{H}_2$$

i.e. bipartite states $\Psi \in \mathcal{H}_1^* \otimes \mathcal{H}_2$ are representable by linear functions $f : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ and vice versa. Indeed

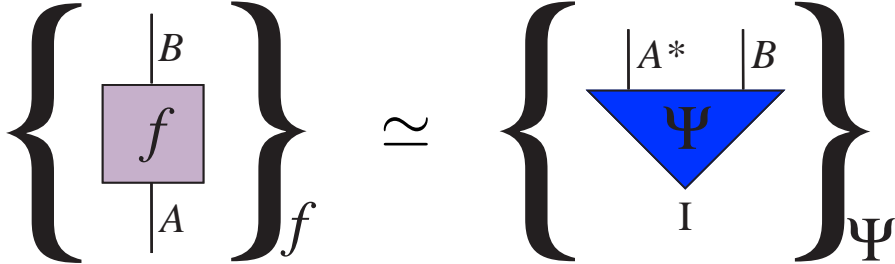
$$\Psi = \sum_{ij} m_{ij} |ij\rangle \quad \xleftrightarrow{\simeq} \quad \begin{pmatrix} m_{11} \cdots m_{1n} \\ \vdots \quad \ddots \quad \vdots \\ m_{k1} \cdots m_{kn} \end{pmatrix}$$

$$\xleftrightarrow{\simeq} \quad f = \sum_{ij} m_{ij} |j\rangle \langle i|$$

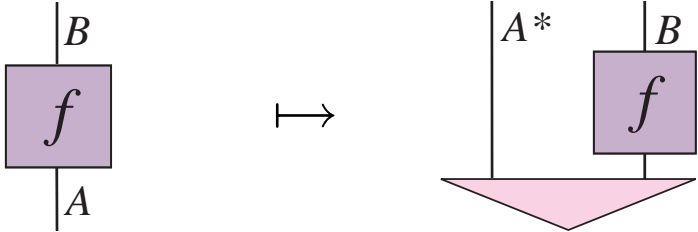
e.g.

$$|00\rangle + |11\rangle \quad \xleftrightarrow{\simeq} \quad \text{id} = |0\rangle \langle 0| + |1\rangle \langle 1|$$

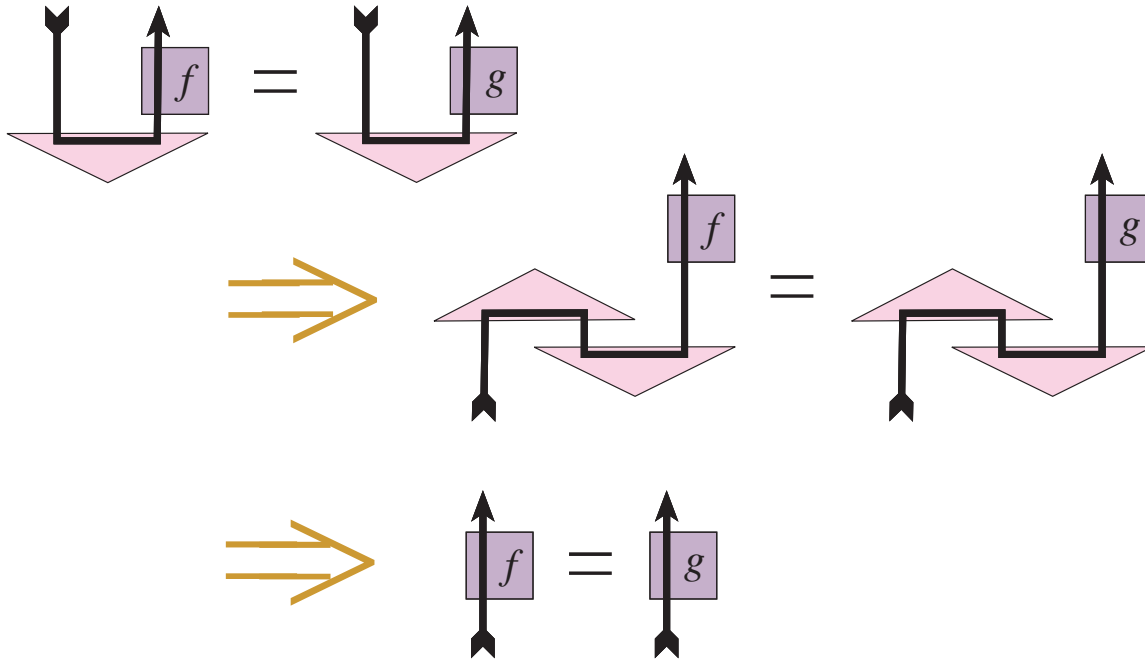
PROCESSES \simeq 2-STATES



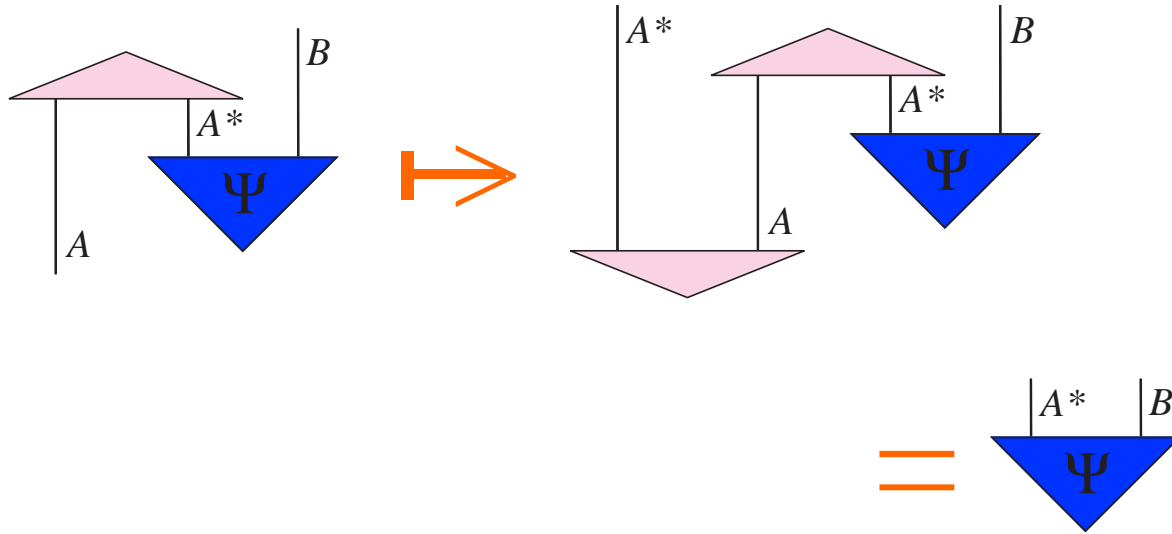
for the bijection $f \mapsto \ulcorner f \urcorner$ i.e.



Proof of injectivity.



Proof of injectivity.



The **inner-product** of $\psi, \phi : I \rightarrow A$ is

$$\langle \phi \mid \psi \rangle := \begin{array}{c} \pi \\ \hline \psi \end{array} = \phi^\dagger \circ \psi : I \rightarrow I$$

where $\pi := \phi^\dagger$ cf.

$$\text{bra} := \langle \phi \mid \quad \text{ket} := \mid \psi \rangle \quad \text{bra-ket} := \langle \phi \mid \psi \rangle$$

e.g. for $f : A \rightarrow B$ we have

$$\mid f \circ \psi \rangle = \begin{array}{c} f \\ \hline \psi \end{array} = f \circ \psi \quad \langle f \circ \phi \mid = \begin{array}{c} \pi \\ \hline f^\dagger \end{array} = \phi^\dagger \circ f^\dagger$$

Adjointness implies

$$\langle f \circ \phi \mid \psi \rangle = \begin{array}{c} \triangle \pi \\ \square f^\dagger \\ \triangle \psi \end{array} = \langle \phi \mid f^\dagger \circ \psi \rangle$$

Unitarity means $U^{-1} = U^\dagger$ i.e.

$$\begin{array}{c} \square U \\ \square U^\dagger \end{array} = \begin{array}{c} \square U^\dagger \\ \square U \end{array} = \left| \right.$$

hence

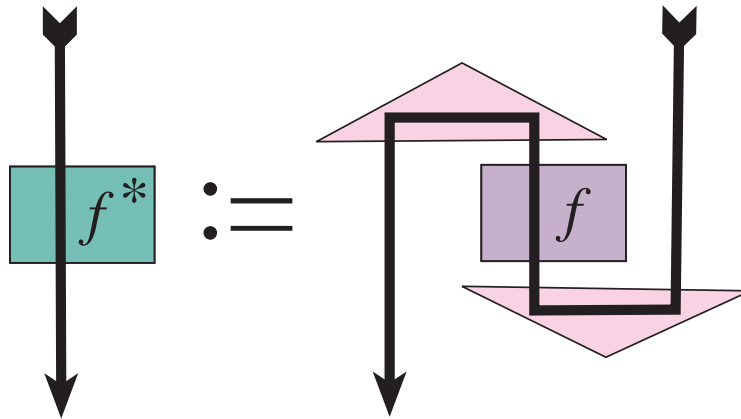
$$\langle U \circ \phi \mid U \circ \psi \rangle = \begin{array}{c} \triangle \pi \\ \square U^\dagger \\ \square U \\ \triangle \psi \end{array} = \begin{array}{c} \triangle \pi \\ \text{---} \\ \triangle \psi \end{array} = \begin{array}{c} \triangle \pi \\ \square \\ \triangle \psi \end{array} = \langle \phi \mid \psi \rangle$$

UPPER STAR STRUCTURE

A “contravariant” Barr-Kelly-Laplaza involution

$$f : A \rightarrow B \quad \mapsto \quad f^* : B^* \rightarrow A^*$$

called **upper star** arises as

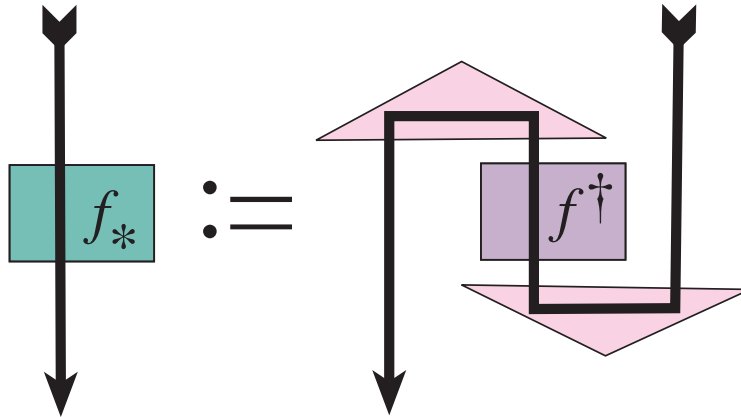


LOWER STAR STRUCTURE

A “covariant” involution

$$f : A \rightarrow B \quad \mapsto \quad f_* : A^* \rightarrow B^*$$

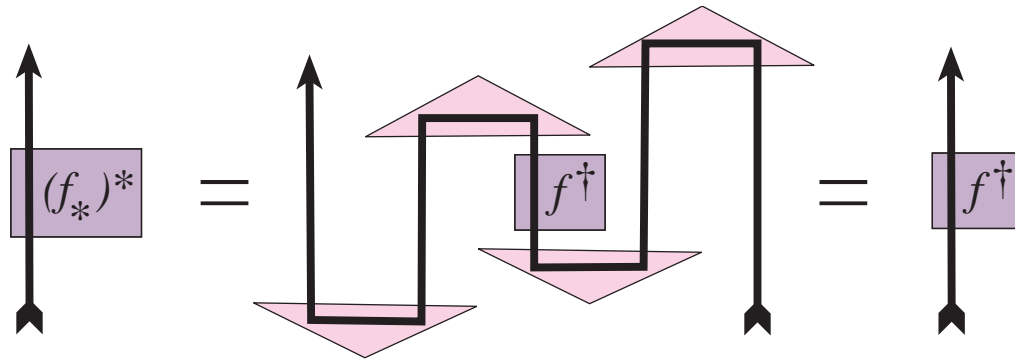
called **lower star** arises as



From



follows



and analogous we can prove that $(f^*)_* = f^\dagger$

Hence the star operations



provide a decomposition of the adjoint:

$$f^\dagger = (f^*)_* = (f_*)^*$$

In particular, for the Hilbert space model we have

$(-)^*$:= **transposition**

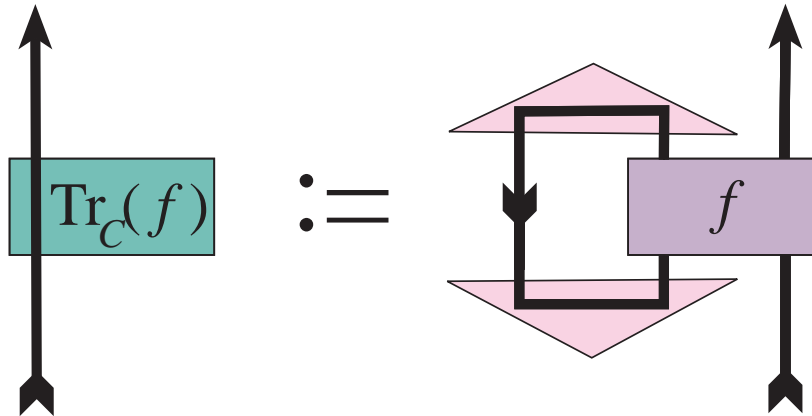
$(-)_*$:= **complex conjugation**

TRACE STRUCTURE

A Joyal-Street-Verity **partial trace**

$$f : C \otimes A \rightarrow C \otimes B \quad \mapsto \quad \text{Tr}_C(f) : A \rightarrow B$$

arises as

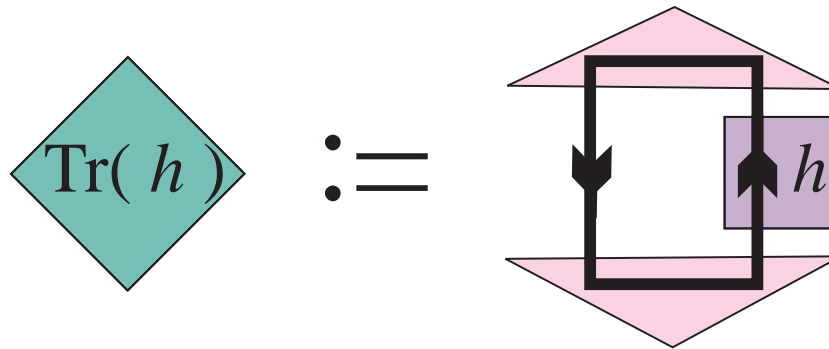


TRACE STRUCTURE bis

A corresponding **full trace**

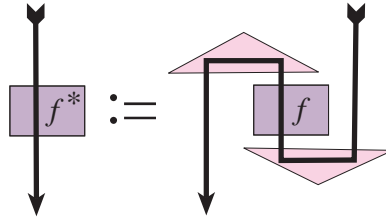
$$h : A \rightarrow A \quad \mapsto \quad \text{Tr}(h) : I \rightarrow I$$

arises as

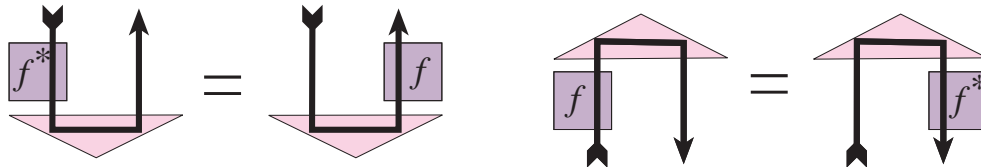


$\Rightarrow h$ “carries a diamond” cf. **probabilistic weight**

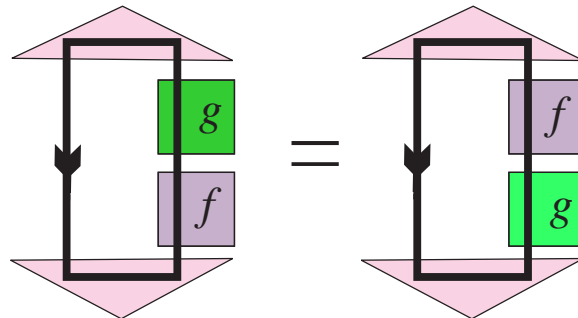
From



follows

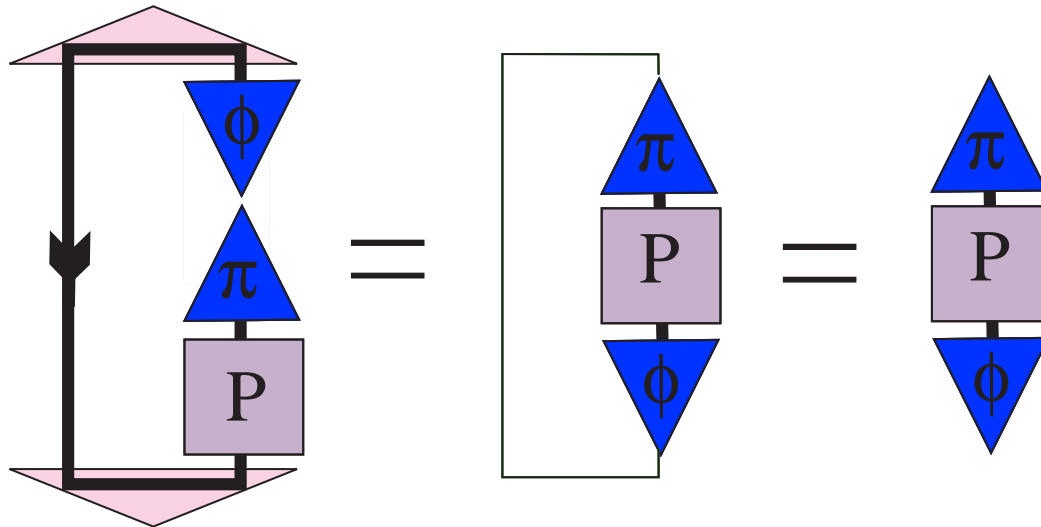


and hence



EQUIVALENT BORN RULES

$$\text{Tr}(\rho_\phi \circ P) \stackrel{???}{=} \langle \phi | P \circ \phi \rangle \quad \text{for} \quad \rho_\phi := |\phi\rangle\langle\phi|$$



$$\mathbb{C}^* \otimes \mathbb{C} \simeq \mathbb{C}$$

ALGEBRA BEHIND THE SCENE

Symmetric monoidal bifunctor $-\otimes- : \mathbf{C} \times \mathbf{C} \rightarrow \mathbf{C}$ and

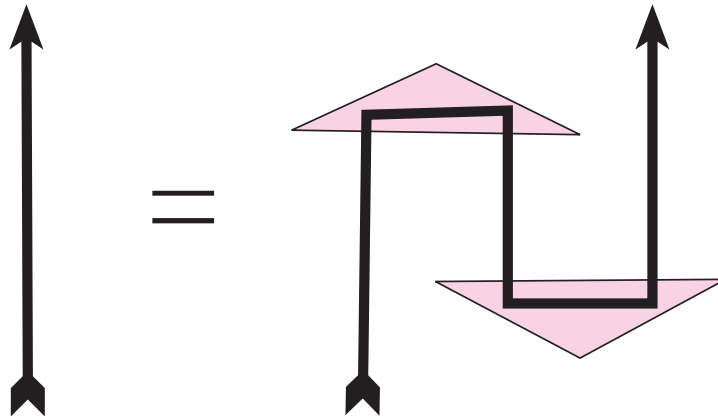
- \otimes -involution **dual** $A \mapsto A^*$;
- contravariant \otimes -involution **adjoint** $f_{A \rightarrow B} \mapsto f_{B \rightarrow A}^\dagger$;
- **Units** $\eta_A : I \rightarrow A^* \otimes A$ with $\eta_{A^*} = \sigma_{A^*, A} \circ \eta_A$;

$$\begin{array}{ccccc}
 A & \xleftarrow{\simeq} & I \otimes A & \xleftarrow{\eta_{A^*}^\dagger \otimes 1_A} & (A \otimes A^*) \otimes A \\
 \uparrow 1_A & & & & \uparrow \simeq \\
 A & \xrightarrow{\simeq} & A \otimes I & \xrightarrow{1_A \otimes \eta_A} & A \otimes (A^* \otimes A)
 \end{array}$$

ALGEBRA BEHIND THE SCENE

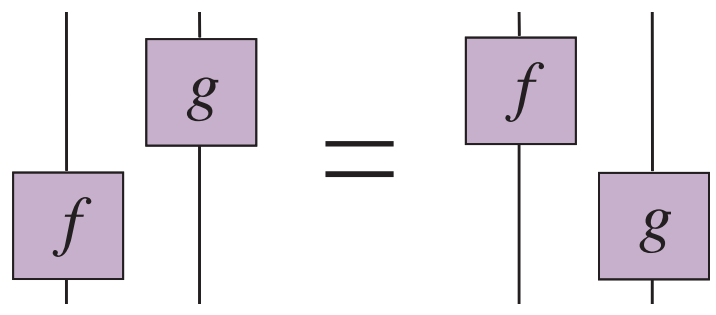
Symmetric monoidal bifunctor $-\otimes- : \mathbf{C} \times \mathbf{C} \rightarrow \mathbf{C}$ and

- \otimes -involution **dual** $A \mapsto A^*$;
- contravariant \otimes -involution **adjoint** $f_{A \rightarrow B} \mapsto f_{B \rightarrow A}^\dagger$;
- **Units** $\eta_A : I \rightarrow A^* \otimes A$ with $\eta_{A^*} = \sigma_{A, A^*} \circ \eta_A$;



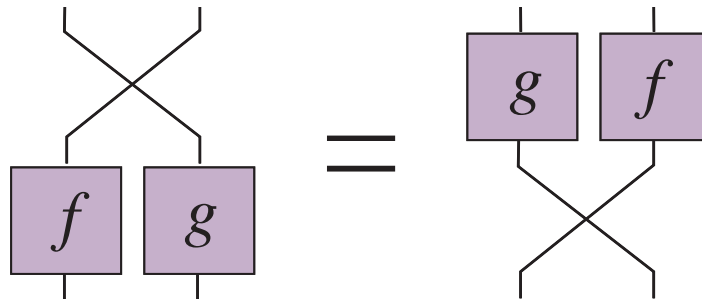
BIFUNCTORIALITY OF \otimes

$$\begin{array}{ccc}
 A_1 \otimes A_2 & \xrightarrow{f_1 \otimes \text{id}} & B_1 \otimes A_2 \\
 \downarrow \text{id} \otimes f_2 & & \downarrow \text{id} \otimes f_2 \\
 A_1 \otimes B_2 & \xrightarrow{f_1 \otimes \text{id}} & B_1 \otimes B_2
 \end{array}$$



NATURAL SYMMETRY

$$\begin{array}{ccc}
 A_1 \otimes A_2 & \xrightarrow{f_1 \otimes f_2} & B_1 \otimes B_2 \\
 \downarrow \sigma_{A_1, A_2} & & \downarrow \sigma_{B_1, B_2} \\
 A_2 \otimes A_1 & \xrightarrow{f_2 \otimes f_1} & B_2 \otimes B_1
 \end{array}$$



STATES AND NUMBERS

We use the unit I for $-\otimes-$ i.e.

$$A \simeq I \otimes A \simeq A \otimes I$$

to define states and numbers respectively as

$$\Psi : I \rightarrow A$$

and

$$s : I \rightarrow I$$

NATURAL SCALAR MULTIPLES

Scalars satisfy

$$s \circ t = I \xrightarrow{\cong} I \otimes I \xrightarrow{s \otimes t} I \otimes I \xrightarrow{\cong} I$$

and we define **scalar multiplication** as

$$s \bullet f := A \xrightarrow{\cong} A \otimes I \xrightarrow{f \otimes s} B \otimes I \xrightarrow{\cong} B$$

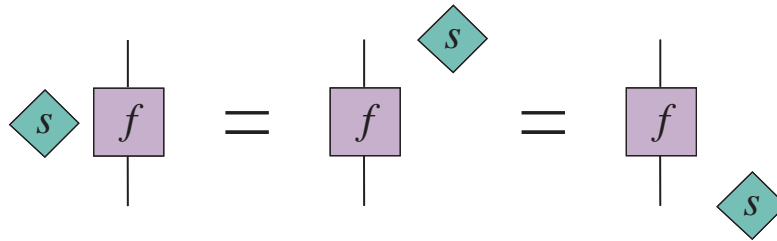
for which we can then prove

$$(s \bullet f) \circ (t \bullet g) = (s \circ t) \bullet (f \circ g)$$

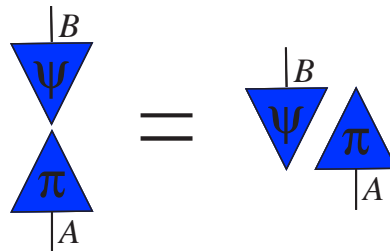
$$(s \bullet f) \otimes (t \bullet g) = (s \circ t) \bullet (f \otimes g)$$

i.e. **diamonds can move around freely in 'time' and 'space'**

NATURAL SCALAR MULTIPLES



and similarly



i.e.

$$\psi \circ \pi = A \xrightarrow{\cong} I \otimes A \xrightarrow{\psi \otimes \pi} B \otimes I \xrightarrow{\cong} B$$

NO-CLONING NO-DELETING

Cf. Dieks-Wooters-Zurek 1982 & Pati-Braunstein 2000

Obviously we do not want to be $-\otimes-$ a **categorical (co-)product** since that would imply existence of

$$A \xrightarrow{\Delta} A \otimes A \qquad A \otimes B \xrightarrow{p} A$$

i.e. there are no logical rules

$$A \vdash A \wedge A \qquad A \wedge B \vdash A$$

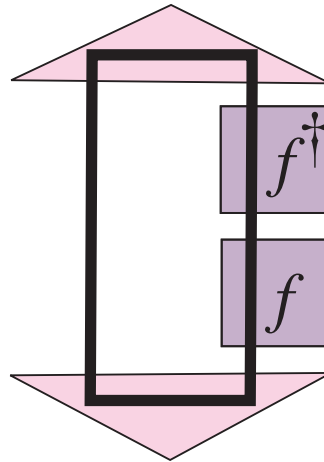
The **squared Hilbert-Schmidt norm**

$$\|f\|^2 = \sum_i \langle f(e_i) | f(e_i) \rangle$$

exists in the picture formalism as

$$\|f\|^2 := (\ulcorner f \urcorner)^\dagger \circ \ulcorner f \urcorner$$

i.e.



The **squared Hilbert-Schmidt norm**

$$\|f\| = \sum_i \langle f(e_i) | f(e_i) \rangle$$

exists in the picture formalism as

$$\|f\| := (\ulcorner f \urcorner)^\dagger \circ \ulcorner f \urcorner$$

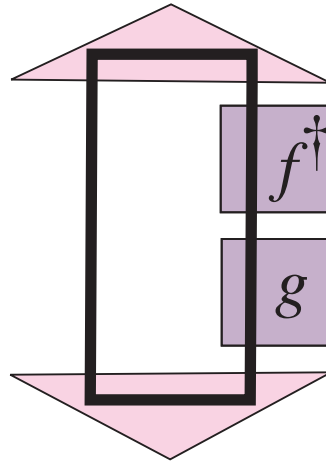
Proof.

$$\begin{aligned} \|f\|(1) &= (\eta^\dagger \circ (1 \otimes f)^\dagger \circ (1 \otimes f) \circ \eta) (1) \\ &= (\eta^\dagger \circ (1 \otimes (f^\dagger \circ f))) \left(\sum e_i \otimes e_i \right) \\ &= \eta^\dagger \left(\sum e_i \otimes f^\dagger(f(e_i)) \right) \\ &= \sum \langle e_i | f^\dagger(f(e_i)) \rangle \\ &= \sum \langle f(e_i) | f(e_i) \rangle. \end{aligned}$$

The corresponding **Hilbert-Schmidt inner-product** also exists in the picture formalism as

$$\langle f | g \rangle := (\ulcorner f \urcorner)^\dagger \circ \ulcorner g \urcorner$$

i.e.



and generalizes 'the one on states' since

$$(\ulcorner \psi \urcorner)^\dagger \circ \ulcorner \phi \urcorner = \psi^\dagger \circ \phi$$

ALL IS QUANTITATIVE!

The **squared Hilbert-Schmidt norm** yields:

a canonical norm on processes

The **Hilbert-Schmidt inner-product** yields:

an inner-product on processes

ABSTRACT GLOBAL PHASES

$$f \otimes f^\dagger = e^{i\theta} \cdot g \otimes (e^{i\theta} \cdot g)^\dagger = e^{i\theta} \cdot g \otimes e^{-i\theta} \cdot g^\dagger = g \otimes g^\dagger$$

Proposition 1.

$$s \bullet f = t \bullet g, s \circ s^\dagger = t \circ t^\dagger = 1_I \implies f \otimes f^\dagger = g \otimes g^\dagger$$

Proposition 2.

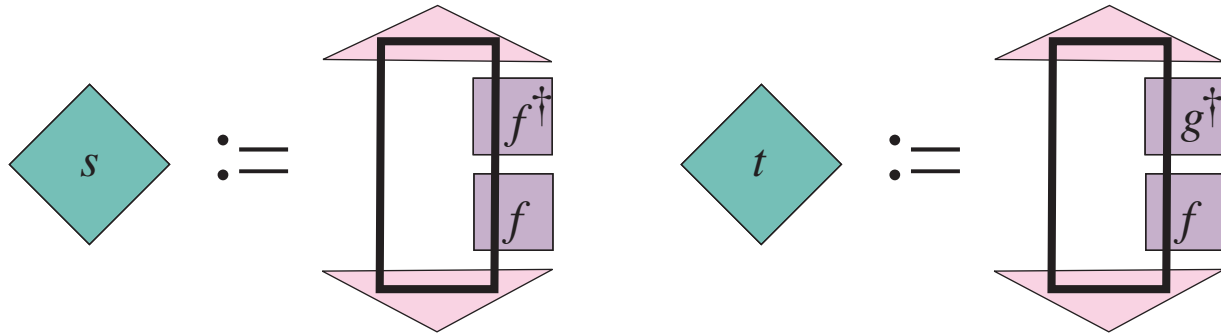
$$f \otimes f^\dagger = g \otimes g^\dagger \implies \exists s, t : s \bullet f = t \bullet g, s \circ s^\dagger = t \circ t^\dagger$$

e.g.

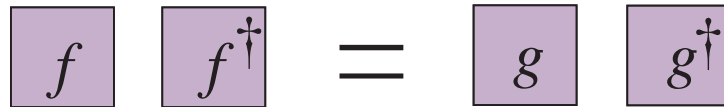
$$s := (\ulcorner f \urcorner)^\dagger \circ \ulcorner f \urcorner \quad \text{and} \quad t := (\ulcorner g \urcorner)^\dagger \circ \ulcorner f \urcorner$$

Proof.

$$\#1 \quad s := (\lceil f \rceil)^\dagger \circ \lceil f \rceil \quad \text{and} \quad t := (\lceil g \rceil)^\dagger \circ \lceil f \rceil$$

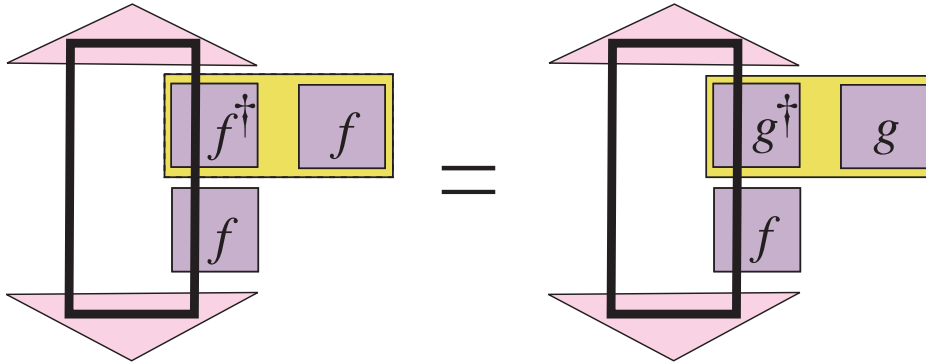


$$\#2 \quad f \otimes f^\dagger = g \otimes g^\dagger$$



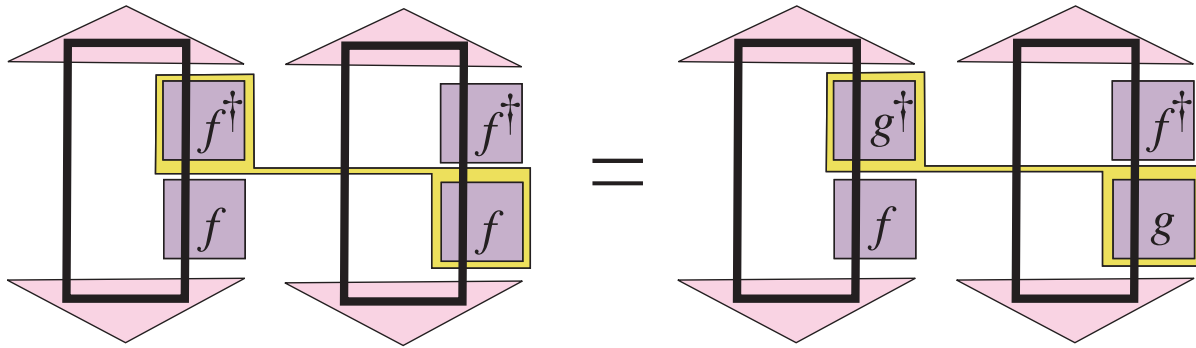
Proof.

$$\#3 \quad s \bullet f = t \bullet g \quad \text{with} \quad s/t := (\lceil f/g \rceil)^\dagger \circ \lceil f \rceil$$

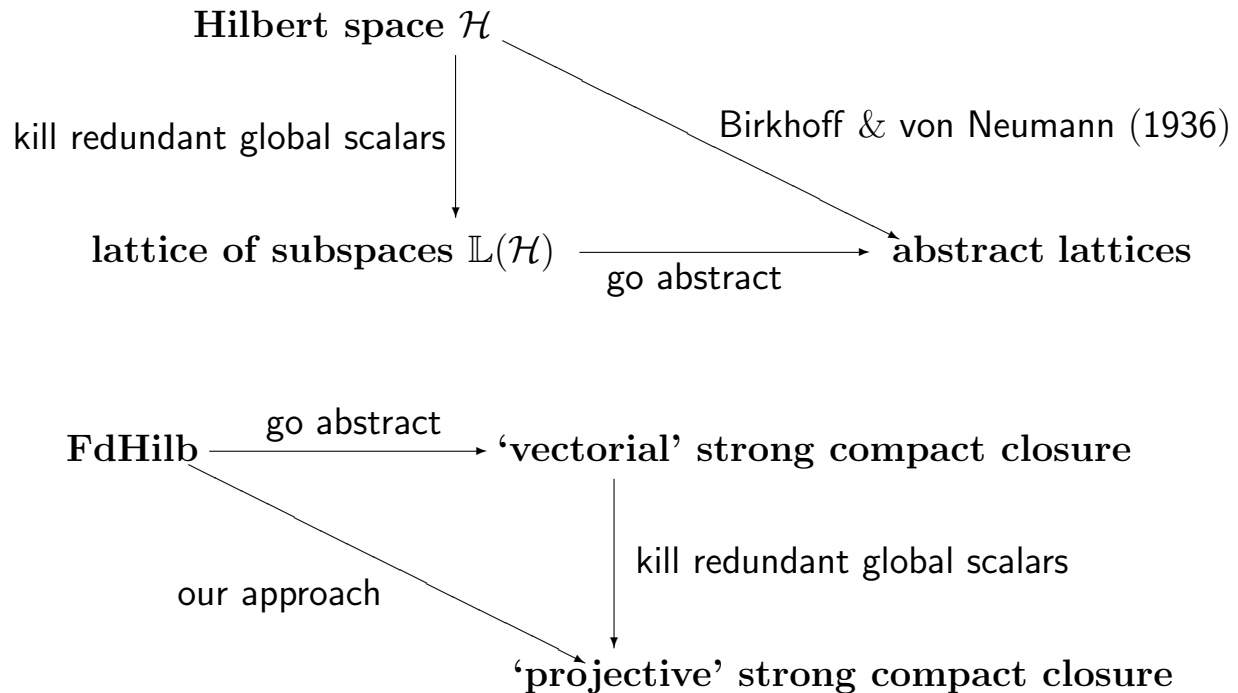


Proof.

$$\#4 \quad s \circ s^\dagger = t \circ t^\dagger \quad \text{with} \quad s/t := (\ulcorner f/g^\urcorner)^\dagger \circ \ulcorner f^\urcorner$$



PROJECTIVE vs VECTORIAL



ABSENCE OF GLOBAL PHASES

Proposition. $WProj(\mathbf{C}) \simeq \mathbf{C}$ (canonically) iff

$$f \otimes f^\dagger = g \otimes g^\dagger \implies f = g$$

iff

$$P_f = P_g \implies \lceil f \rceil = \lceil g \rceil$$

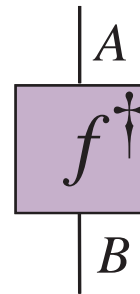
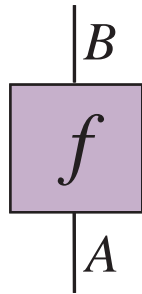
iff

$$\psi \circ \psi^\dagger = \phi \circ \phi^\dagger \implies \psi = \phi$$

iff

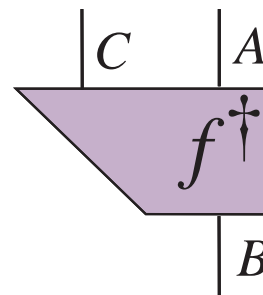
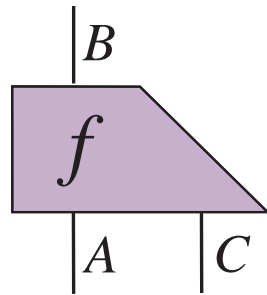
Equal Preparations Produce Equal States

OPEN SYSTEMS AND CPMs



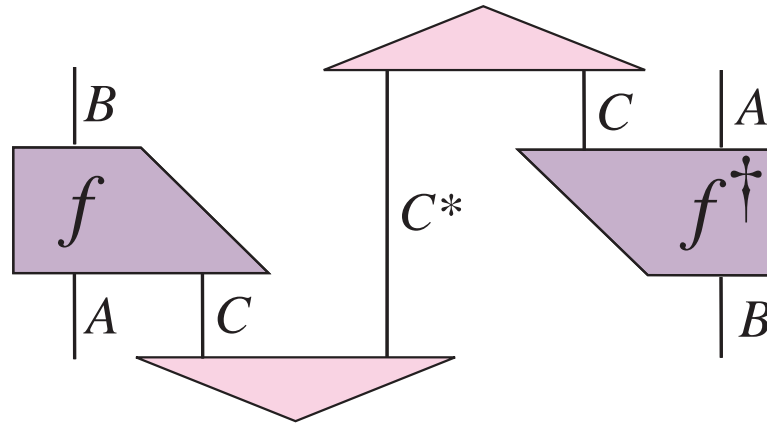
\Rightarrow projective process

OPEN SYSTEMS AND CPMs



\Rightarrow projective process with ancilla

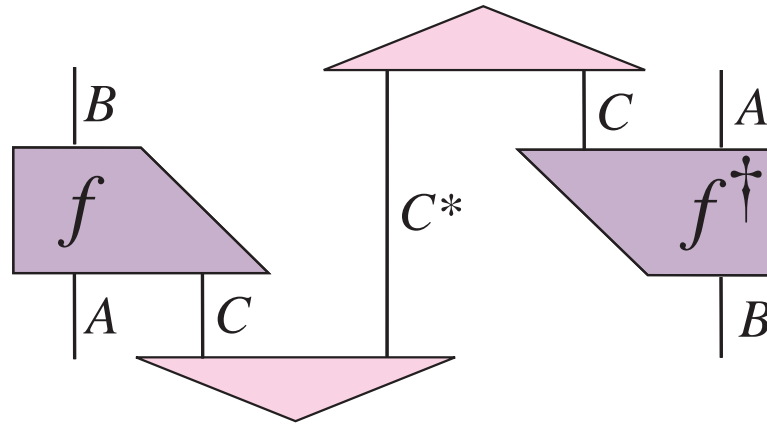
OPEN SYSTEMS AND CPMs



\Rightarrow projective process with hidden ancilla

= open process on open system

OPEN SYSTEMS AND CPMs



In the case of **Hilbert spaces** and **linear maps** we exactly obtain **completely positive maps** (Selinger 2005)!

ABSTRACT QM

System of type A := Object A

Composite of A and B := Tensor $A \otimes B$

Process of type $A \rightarrow B$:= Morphism $f : A \rightarrow B$

State of A := Element $\psi : I \rightarrow A$

Evolution of A := Unitary $U : A \rightarrow A$

Measurement on A := “Projectors” $\{P_i : A \rightarrow A\}_i$

- Data := $\nu \in \{i\}_i$
- Dynamics := $\psi \mapsto P_\nu \circ \psi$
- Probability := $\psi^\dagger \circ P_\nu \circ \psi = \text{Tr}(P_\nu \circ \rho_\psi) : I \rightarrow I$

Some extra structure is required both for

- Specification of the families $\{P_i : A \rightarrow A\}_i$
- Combining $\{P_i\}_i$ into a single $M : A \rightarrow \dots$

But, **you can pick your favorite!**

For each unitary morphism $U : A \rightarrow \bigoplus_i A_i$ we have

$$\{P_j := \pi_j^\dagger \circ \pi_j\}_j \quad M := \left(\bigoplus_i \pi_i^\dagger \right) \circ U : A \rightarrow \bigoplus_i A$$

where $\pi_j := p_j \circ U$. Alternatively, $\{f_i\}_i$ has to satisfy $\sum_i f_i = 1_A$ and the corresponding measurement is

$$M := \langle f_i \rangle_i : A \rightarrow \bigoplus_i A.$$

DIGEST

- ... first full formal description of protocols
- ... types reflect kinds
- ... classical data-flow is included
- ... quantum info-flow is explicit
- ... kindergarten description/correctness proofs
- ... space for formal/conceptual choices
- ... the thing people call QM-relationalism?

APPLICATIONS

— “why computer scientists care about this stuff” —

Quantum programming language design

Quantum program logics for verification

Quantum protocol specification

Quantum protocol verification

Appropriate semantics for new quantum computational paradigms e.g. one-way (Briegel), teleportation based (Gottesman-Chuang), measurement based in general, topological quantum computing (Kitaev et al.) etc.

RELATED WORK

Penrose. *Applications of negative dimensional tensors* (1971) \Rightarrow Diagrammatic reasoning in physics (GR)

Kauffman. *Teleportation topology*. quant-ph/0407224 \Rightarrow Independent logic of entanglement observation as Coecke PRG-R-03-12 & quant-ph/0402014

Baez. *Quantum quandaries: a category-theoretic perspective*. quant-ph/0404040 \Rightarrow Independent Rel-QM connection observation as Abramsky-Coecke quant-ph/0402130; also, GR-QM structural connection

Deligne. *Catégories tannakiennes*. In *The Grothendieck Festschrift* (1990). \Rightarrow Representation Theorem !!!